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DEPARTMENT OF ECONOMICS  
RESEARCH MEMORANDUM



**BASICS OF INVENTORY MANAGEMENT: PART 5**  
**The (R,b,Q)-model**

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BASICS OF INVENTORY MANAGEMENT: PART 5.

The  $(R,b,Q)$ -model.

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## BASICS OF INVENTORY MANAGEMENT: INTRODUCTION

In the winter of 1989 the idea emerged to document the knowledge about inventory management models, that had been developed over almost 10 years of research and 5 years of practical applications in a number of consultancy projects. The main motivation to document the methodology underlying a number of well-proven algorithms was that most existing literature did not cover the practical applications encountered. Investigations revealed that most well-known algorithms were based on the assumptions of stable demand during lead times and large batch sizes. Both assumptions do not apply to the JIT environment characterized by short lead times and high order frequencies.

My starting point was the application of renewal theory to production-inventory models. It turned out that the same formalism was applicable to the classical inventory models, like periodic review and reorder point models. The attention of the analysis was focused on service levels and average inventories. The reason for this was that in many cases the problem was to find a relation between customer service requirements and holding costs for different planning scenarios. The algorithms developed turned out to be robust and fast.

The conviction grew that the methodology extended to most practically relevant service measures and to all classical inventory models. To be able to prove this sponsors were needed to provide the time and money to do the required research. The Catholic University Brabant and the Centre for Quantitative Methods accepted the research proposal. The result of the research is the series **Basics of Inventory Management**.

From the outset the objective was to develop a unified framework for all classical inventory models. It was important to relax a number of assumptions made in most literature. To the knowledge of the author for the first time arbitrary compound renewal demand processes are considered, thereby relaxing the assumption of Poisson customer arrival processes. This is very important in view of market concentrations (hyper markets, power retailers,



etc.). The outcome of the research should be a comprehensive set of algorithms, which can be used in practical situations, e.g. in inventory management modules of MRP and DRP packages.

In the course of the research the so-called PDF-method was developed, that provided a means to approximately solve all relevant mathematical equations derived in the analysis. The results of the approximation schemes were promising, yet under some conditions the performance was not adequate. Coincidentally, it turned out that the performance of the PDF-method deteriorated as the order batch size increased. In the area of large batch sizes other approximation schemes had already been developed, so that together with the PDF-method these algorithms covered the whole range of models.

Though starting from the idea to provide practically useful material to OR-practitioners, it soon turned out that the analysis required was quite detailed and mathematically intricate. Nonetheless I felt it necessary to document the derivations as well, since the analysis extends to other models than discussed in this series. The consequence of this choice is that the first 6 parts (c.q chapters) of this series are entirely mathematical. Yet the reader will find as a result of the analysis simple-to-use approximation schemes. To illustrate the applicability of the analysis, part VII is devoted to numerical analysis, part VIII compares the different inventory management models and part IX provides a number of practical cases.

Part I provides the background material from renewal theory and the PDF-method. Part II discusses the  $(R,S)$ -model, part III the  $(b,Q)$ -model and part IV the cost-optimal  $(s,S)$ -model. Based on the analysis in part II-IV we analyze in part V and VI the  $(R,b,Q)$ - and the  $(R,s,S)$ -model, respectively. A provisional list of references is given below.

I would like to thank Frank van der Duyn Schouten of the Catholic University Brabant for giving me the funds to do the research. The same holds for Jos de Kroon and Mynt Zijlstra from the Centre for Quantitative Methods of Philips. Furthermore, I would like to thank Marc Aarts and Jan-Maarten van Sonsbeek for programming work.

THE (R,b,Q)-MODEL

The (R,b,Q)-strategy applies to situations where decisions are made periodically, once a week, say and order procurement costs are too high to allow for an (R,S)-strategy. The (R,b,Q)-policy is applied implicitly in many MRP-packages, where fixed lot sizes are used and a time phased order point determines the order (or explosion) moments.

The (R,b,Q)-strategy is described as follows:

*Stock is reviewed every  $R^{\text{th}}$  time unit. If at a review moment the inventory position is below  $b$ , then an integral multiple of  $Q$  is ordered, such that the inventory position is raised to a value between  $b$  and  $b+Q$ .*

The analysis of the (R,b,Q)-model is quite similar to the analysis of the (b,Q)-model. This chapter is organized as follows. In section 6.1. we describe the model in more detail. In section 6.2. expressions for the  $P_2$ -measure and the fill rate are derived. In section 6.3. we discuss the mean physical stock and the mean backlog.

6.1. Model description

We consider two instances of the (R,b,Q)-model. First we describe the discrete time situation, where depletion of stock is registered at equidistant points in time and secondly, we describe the situation, where depletion of stock is registered after each customer arrival. The latter system is a so-called real time inventory management system, the former system operates in a batch-mode.

I: The discrete time situation

We agree upon a time unit, a day, say, at the end of which we collect data about stock depletion during the time unit, as well as arrivals of replenishments during that time unit. Next we decide about the review period, i.e. how many time units elapse

between decision epochs, at which we may order an amount at the supplier. Let  $R$  be the review period duration,  $R$  is an integral number of time units. Then decisions about when and how much to order are governed by the  $(R,b,Q)$ -policy.

Due to the fact that during a time unit replenishments may arrive, while stock is also depleted, we must agree upon the way we define disservice and shortages. Indeed, it differs if the replenishment arrives at the beginning of the time unit or at the end of it. We assume the following pessimistic way of processing the data about replenishments and stock depletions.

*We assume that a replenishment arrives at the end of a time unit.*

As in chapter 3 we describe the demand process by  $\{D_n\}$ , with

$D_n$  := demand during time unit  $n$ .

$\{D_n\}$  is a sequence of i.i.d. random variables. Furthermore we have a sequence of lead times  $\{L_k\}$ , which are identically distributed and are such that orders cannot overtake. Each lead time is an integral number of time units.

## II: The compound renewal situation

In this case we assume that customers arrive according to a compound renewal demand process. The sequence of interarrival times  $\{A_n\}$  form a renewal process. The same holds for the demands per customer  $\{D_n\}$ . The lead times  $\{L_k\}$  are identically distributed and orders cannot overtake. In this case we do not encounter problems concerning the processing of inventory transactions, since each transaction is processed individually.

### 6.2. The service measures

We want to determine the reorder level  $b$ , such that for a given value of  $Q$  a target service level is achieved. As before we restrict ourselves to the  $P_2$ -measure and the  $\hat{P}_1$ -measure.



## P<sub>2</sub>-measure

We derive an expression for the P<sub>2</sub>-measure for any demand process. Consider an order cycle, i.e. the time between two consecutive order moments. We define the random variables  $\sigma_1$ ,  $D_R$  and  $U_{R,i}$ ,  $i=0,1$  as:

$\sigma_1$  := the point in time at which the inventory position drops below  $b$  for the first time after time 0.

$D_R$  := demand during  $(0, R]$ .

$U_{R,0}$  := the undershoot of  $b$  at time 0.

$U_{R,1}$  := the undershoot of  $b$  at time  $\sigma_1$ .

Then

$$D_R = \sum_{n=1}^R D_n \quad (6.1)$$

$$\begin{aligned} D(0, \sigma_1] &= b + Q - U_{R,0} - (b - U_{R,1}) \\ &= Q - U_{R,0} + U_{R,1} \end{aligned} \quad (6.2)$$

Note that we implicitly assume that only one batch of size  $Q$  is ordered. Therefore we must assume that

$$Q \gg E[D_R].$$

It turns out that the results derived even hold for  $Q \approx E[D_R]$ , yet from a mathematical point of view the above assumption is necessary.

When we compare the evolution of the inventory position for the  $(R, b, Q)$ -model with that for the  $(b, Q)$ -model we see that  $D_R$  in the  $(R, b, Q)$ -model plays the role of the demand per customer in the

(b,Q)-model. Then we apply the approximation for the undershoot in the (b,Q)-model to  $U_{R,i}$ ,

$$P\{U_{R,i} \leq x\} = \frac{1}{E[D_R]} \int_0^x (1 - F_{D_R}(y)) dy \quad (6.3)$$

Next we consider the replenishment cycle  $(L_0, \sigma_1 + L_1]$ , where

$L_0$  := lead time of order initiated at time 0.

$L_1$  := lead time of order initiated at time  $\sigma_1$ .

As for the (b,Q)-model we can derive the following expression for the  $P_2$ -measure:

$$P_2(b, Q) = 1 - \frac{\{E[(D_{(\sigma_1, \sigma_1 + L_1)} + U_{R,1} - b)^+] - E[(D_{(0, L_0)} + U_{R,0} - (b, Q))^+]\}}{E[D(0, \sigma_1)]} \quad (6.4)$$

Since (6.4) is identical to (4.4) we can apply all the results in section (4.1) in order to obtain an expression for  $P_2(b, Q)$ , which is based on the PDF-method. Without going into further detail we claim that

$$P_2(b, Q) \approx \hat{\gamma}(b, Q), \quad (6.5)$$

where  $\hat{\gamma}$  is the gamma distribution with its first two moments  $E[X_\gamma]$  and  $E[X_\gamma^2]$  given by

$$E[X_\gamma] = E[D(0, L_0) + U_{R,0}] + \frac{1}{2}Q \quad (6.6)$$

$$E[X_\gamma^2] = E[(D(0, L_0) + U_{R,0})^2] + Q E[D(0, L_0) + U_{R,0}] + \frac{Q^3}{3} \quad (6.7)$$

Equations (6.6) and (6.7) are the equivalent of (4.7) and (4.8), respectively. From (6.1) and (6.3) we derive that

$$E[U_{R,0}] = \frac{E[D_R^2]}{2E[D]} \quad (6.8)$$

$$E[U_{R,0}^2] = \frac{E[D_R^3]}{3E[D]} \quad (6.9)$$

Now we distinguish between the discrete time case and the compound renewal case.

#### Case I: Discrete time case

We assume that  $D_R$  is gamma distributed. This yields

$$E[D_R] = R E[D] \quad (6.10)$$

$$E[D_R^2] = R \sigma^2(D) + R^2 E^2[D] \quad (6.11)$$

$$E[D_R^3] = (1 + C_{D_e}^2) (1 + 2C_{D_e}^2) E^3[D_R], \quad (6.12)$$

with  $C_{D_R}$  the coefficient of variation of  $D_R$ , which can be derived from (6.10) and (6.11).

The first two moments of  $D(0, L_0)$  are given by equations (3.5) and (3.6), which are repeated below.

$$E[D(0, L_0)] = E[L] E[D] \quad (6.13)$$

$$E[D^2(0, L_0)] = E[L] \sigma^2(D) + E[L^2] E^2[D] \quad (6.14)$$

Case II: Compound renewal case

We again assume that  $D_R$  is gamma distributed, such that (6.12) holds. As in section 3.2.2. we make the following assumption about review moments and replenishment moments.

*From the point of view of the arrival process, the review moments and replenishment moments are arbitrary points in time.*

Then we can apply (3.43) and (3.44) to yield

$$E[D_R] \approx \frac{R}{E[A]} E[D] \quad (6.15)$$

$$E[D_R^2] \approx \left[ \frac{R^2}{E^2[A]} + \frac{R}{E[A]} (C_A^2 + C_D^2) + \frac{1 - C_A^4}{16} \right] E^2[D] \quad (6.16)$$

$$E[D(0, L_0)] \approx \frac{E[L]}{E[A]} E[D] \quad (6.17)$$

$$E[D^2(0, L_0)] \approx \left[ \frac{E[L^2]}{E^2[A]} + \frac{E[L]}{E[A]} (C_A^2 + C_D^2) + \frac{(1 - C_A^4)}{16} \right] E^2[D] \quad (6.18)$$

For both cases we have the required expressions to calculate  $E[X_r]$  and  $E[X_r^2]$  and it is routine to apply the PDF-method.



# $\hat{P}_1$ -measure

The  $\hat{P}_1$ -measure yields more complicated mathematics than the  $P_2$ -measure as the reader must have noticed in the preceding chapters. We need to have a close look at the demand process and the evolution of the net stock in time. We immediately must distinguish between the different demand processes described in section 6.1. We first consider the discrete time model.

## Case I: The discrete time model

To obtain results for the  $\hat{P}_1$ -measure in this case we proceed similar to the analysis preceding equation (3.19) for the mean physical stock. In chapter 2 we defined the function  $T^+(x,t)$  by

$T^+(x,t) \quad := \quad$  the expected time the net stock is positive during  $(0,t]$ , given the net stock at time 0 is  $x \geq 0$ .

Then equation (2.51) tells us that

$$E[T^+(x,t)] = M(x) - \int_0^x M(x-y) dF_{D(0,t]}(y) \quad (6.19)$$

The net stock at the beginning of replenishment cycle  $(L_0, \sigma_1 + L_1]$  equals  $b+Q-U_{0,R}-D(0,L_0]$ . Conditioning on the net stock at time  $L_0$  we find

$$\begin{aligned} E[T^+(b,Q)] &= \int_0^{b+Q} M(b+Q-y) dF_{U_{0,R}+D(0,L_0]}(y) \\ &\quad - \int_0^{b+Q} M(b+Q-y) dF_{U_{0,R}+D(0,\sigma_1+L_1]}(y) \end{aligned}$$

Since

$$U_{0,R} + D(0, \sigma_1 + L_1] = Q + U_{1,R} + D(\sigma_1, \sigma_1 + L_1]$$

we find

$$\begin{aligned} E[T^+(b, Q)] &= \int_0^{b+Q} M(b+Q-y) dF_{U_{1,x}+D(0,L_1]}(y) \\ &\quad - \int_0^b M(b-y) dF_{U_{1,x}+D(\sigma_1,\sigma_1+L_1]}(y) \end{aligned} \quad (6.20)$$

Let us take a close look at the time interval  $(\sigma_1 - R_1, \sigma_1]$ . At some time  $\sigma_1 + T_U - R$  in  $(\sigma_1 - R_1, \sigma_1]$  the inventory position drops below by an amount  $U_1$ , say. Then it is clear that

$$U_{1,R} = U_1 + \sum_{n=T_U+1}^R D_n \quad (6.21)$$

The undershoot  $U_1$  is the undershoot in the continuous review  $(b, Q)$ -model with demand per customer  $D_n$ . Hence

$$P\{U_1 \leq x\} \approx \frac{1}{E[D]} \int_0^x (1 - F_D(y)) dy \quad (6.22)$$

Furthermore it can be shown that

$$P\{T_U = t\} = \frac{1}{R} \quad t=1, \dots, R, \quad (6.23)$$

which is intuitively appealing. Equation (6.23) tells us that the level  $b$  is undershot at any time in  $(\sigma_1 - R, \sigma_1]$  with equal probability.

Define the random variable  $W$  by

$$W := \sum_{n=T_0+1}^R D_n$$

Then (6.20) can be rewritten as

$$\begin{aligned} E[T^*(b, Q)] &= \int_0^{b+Q} \int_0^{b+Q-y} M(b+Q-y-z) dF_U(z) dF_{W+D(0, L_0)}(y) \\ &\quad - \int_0^b \int_0^{b-y} M(b-y-z) dF_U(z) dF_{W+D(0, L_0)}(y) \end{aligned}$$

Now we apply the identity

$$\int_0^x M(x-y) dF_U(y) = \frac{x}{E[D]} \quad x \geq 0,$$

with U distributed according to (6.22) to obtain

$$\begin{aligned} E[T^*(b, Q)] &= \int_0^{b+Q} \frac{(b+Q-y)}{E[D]} dF_{W+D(0, L_0)}(y) \\ &\quad - \int_0^b \frac{(b-y)}{E[D]} dF_{W+D(0, L_0)}(y) \\ &= \frac{Q}{E[D]} - \frac{1}{E[D]} \left[ \int_b^\infty (y-b) dF_{W+D(0, L_0)}(y) \right. \\ &\quad \left. - \int_{b+Q}^\infty (y-(b+Q)) dF_{W+D(0, L_0)}(y) \right] \end{aligned} \tag{6.24}$$

By definition we have that

$$\hat{P}_1(b, Q) = \frac{E[T^*(b, Q)]}{E[\sigma_1]}$$

Thus we find the following expression for  $\hat{P}_1(b, Q)$ ,

$$\hat{P}_1(b, Q) = 1 - \frac{1}{Q} \left[ \int_b^{\infty} (y-b) dF_{W+D(0, L_0)}(y) - \int_{b+Q}^{\infty} (y-(b+Q)) dF_{W+D(0, L_0)}(y) \right] \quad (6.25)$$

We can alternatively write (6.25) as

$$\hat{P}_1(b, Q) = 1 - \frac{E[(W+D(0, L_0) - b)^+] - E[(W+D(0, L_0) - (b+Q))^+]}{Q}$$

Note the remarkable resemblance of the above equation with equation (6.4) for the  $P_2$ -measure. Therefore we can proceed along the same lines as in the derivation of the first two moments of the gamma fit of  $P_2(b, Q)$ .

So let  $X_\gamma$  be the random variable associated with  $\hat{P}_1(b, Q)$ . Then we have

$$E[X_\gamma] = E[D(0, L_0) + W] + \frac{1}{2}Q \quad (6.26)$$

$$E[X_\gamma^2] = E[(D(0, L_0) + W)^2] + Q E[D(0, L_0) + W] + \frac{Q^2}{3} \quad (6.27)$$

It remains to find an expression for the first two moments of  $W$ . Recall that

$$W = \sum_{n=T_0+1}^R D_n$$

Since  $\{D_n\}$  independent of  $T_u$  this yields



$$E[W] = (R - E[T_U]) E[D] \quad (6.28)$$

$$E[W^2] = (R - E[T_U]) \sigma^2(D) + (R^2 - 2RE[T_U] + E[T_U^2]) E^2[D] \quad (6.29)$$

The problem of finding  $E[W]$  and  $E[W^2]$  has been reduced to finding  $E[T_U]$  and  $E[T_U^2]$ . These follow from (6.23).

$$E[T_U] = \frac{(R+1)}{2} \quad (6.30)$$

$$E[T_U^2] = \frac{1}{6} R(2R+1) \quad (6.31)$$

Equations (6.26) - (6.31) enable us to compute  $E[X_\gamma]$  and  $E[X_\gamma^2]$ . Fitting a gamma distributed  $\hat{\gamma}(\cdot)$  to  $P_1(b, Q)$  we find

$$\hat{P}_1(b, Q) \approx \hat{\gamma}(b+Q) \quad b \geq -Q$$

and the service level equation

$$\hat{P}_1(b^*, Q) = \alpha$$

can be approximately solved by

$$b^* \approx \hat{\gamma}^{-1}(\alpha) - Q$$

This concludes the analysis of the discrete time model.

Case II: The compound renewal demand model

As in the discrete time case we start with an approximation for  $E[T^+(x, t)]$  derived in chapter 2. Equation (2.53) states that in the compound renewal case

$$E[T^+(x, t)] \approx (E[\tilde{A}] - E[A]) (1 - F_{D(0, t]}(x)) + E[A] (M(x) - \int_0^x M(x-y) dF_{D(0, t]}(y))$$

The net stock at the beginning of replenishment cycle  $(L_0, \sigma_1 + L_1]$  equals again  $b + Q - U_{0,R} - D(0, L_0]$  and therefore we find

$$E[T^+(b, Q)] \approx (E[\tilde{A}] - E[A]) (F_{D(0, L_0] + U_{0,R}}(b + Q) - F_{D(0, \sigma_1 + L_1] + U_{0,R}}(b, Q) + E[A] \left[ \int_0^{b+Q} M(b+Q-y) dF_{D(0, L_0] + U_{0,R}}(y) - \int_0^{b+Q} M(b+Q-y) dF_{D(0, \sigma_1 + L_1] + U_{0,R}}(y) \right])$$

Since

$$P\{D(0, \sigma_1 + L_1] + U_{0,R} \leq x\} = P\{D(\sigma_1, \sigma_1 + L_1] + U_{1,R} \leq x - Q\} \quad x \geq Q$$

and  $D(\sigma_1, \sigma_1 + L_1] + U_{1,R}$  is identically distributed to  $D(0, L_0] + U_{0,R}$ , we find

$$\begin{aligned}
 E[T^*(b, Q)] &\approx (E[\tilde{A}] - E[A]) (F_{D(0, L_0] + U_{0,R}}(b+Q) - F_{D(0, L_0] + U_{0,R}}(b)) \\
 &+ E[A] \left[ \int_0^{b+Q} M(b+Q-y) dF_{D(0, L_0] + U_{0,R}}(y) \right. \\
 &\quad \left. - \int_0^b M(b-y) dF_{D(0, L_0] + U_{0,R}}(y) \right]
 \end{aligned} \tag{6.32}$$

As in the discrete time case we express the periodic review undershoot  $U_{0,R}$  in terms of the customer undershoot of level  $b$ ,  $U_0$ . Towards this end we define

$T_U :=$  the time at which the level  $b$  is undershoot by the demand of a customer,  $\sigma_1 - R \leq T_U \leq \sigma_1$ .

We conjecture the following for  $Q$  sufficiently large.

$$P\{T_U \leq t\} = \frac{t}{R} \quad 0 \leq t \leq R \tag{6.33}$$

$T_U$  and  $U_0$  are independent.

It can be shown that this conjecture holds asymptotically for  $Q \rightarrow \infty$  and compound Poisson demand. For arbitrary arrival processes the conjecture was verified empirically by computer simulation. Define  $N(t)$  by

$N(t) :=$  the number of customers arriving in  $(0, t]$ , given that at time  $t$  a customer arrived.

Then we have the following relation between  $U_{0,R}$  and  $U_0$ ,

$$U_{0,R} = U_0 + \sum_{n=1}^{N(R-T_U)} D_n \tag{6.34}$$

Define the random variable  $W$  as

$$W := \sum_{n=1}^{N(R-T_0)} D_n$$

Then it follows that

$$D(0, L_0] + U_{0,R} = U_0 + W + D(0, L_0]$$

Convolving  $M(\cdot)$  with  $U_0$  in (6.32) yields

$$\begin{aligned} E[T^+(b, Q)] &\approx (E[\tilde{A}] - E[A]) (F_{D(0, L_0] + U_{0,R}}(b+Q) - F_{D(0, L_0] + U_{0,R}}(b)) \\ &+ E[A] \left[ \int_0^{b+Q} \frac{(b+Q-y)}{E[D]} dF_{W+D(0, L_0]}(y) \right. \\ &\quad \left. - \int_0^b \frac{(b-y)}{E[D]} dF_{W+D(0, L_0]}(y) \right], \end{aligned}$$

which can be rewritten into

$$\begin{aligned} E[T^+(b, Q)] &\approx \frac{(c_A^2 - 1)}{2} E[A] (F_{D(0, L_0] + U_{0,R}}(b+Q) - F_{D(0, L_0] + U_{0,R}}(b)) \\ &+ E[A] \left( \frac{Q}{E[D]} - \left[ \int_b^\infty \frac{(y-b)}{E[D]} dF_{W+D(0, L_0]}(y) \right. \right. \\ &\quad \left. \left. - \int_{b+Q}^\infty (y - (b+Q)) dF_{W+D(0, L_0]}(y) \right] \right) \end{aligned} \quad (6.35)$$

Dividing (6.35) by  $E[\sigma_1]$  we find



$$\begin{aligned} \hat{P}_1(b, Q) &\approx \frac{(c_A^2-1)}{2} \frac{E[D]}{Q} (F_{D(0, L_0)+U_{0,x}}(b+Q) - F_{D(0, L_0)+U_{0,x}}(b)) \\ &+ 1 - \frac{1}{Q} \left[ \int_b^\infty (y-b) dF_{W+D(0, L_0]}(y) - \int_{b+Q}^\infty (y-(b+Q)) dF_{W+D(0, L_0]}(y) \right] \end{aligned} \quad (6.36)$$

Equation (6.36) is well suited for application of the PDF-method. Applying by now standard arguments we find

$$\begin{aligned} E[X_\gamma] &= \int_{-Q}^\infty (1 - \hat{P}_1(x, Q)) dx \\ &= \frac{(c_A^2-1)}{2} E[D] + E[W+D(0, L_0)] + \frac{Q}{2} \end{aligned} \quad (6.37)$$

$$\begin{aligned} E[X_\gamma^2] &= 2 \int_{-Q}^\infty (x+Q) (1 - \hat{P}_1(x, Q)) dx \\ &= -\frac{(c_A^2-1)}{2} (Q+2E[D(0, L_0]+U_{0,R}]) \\ &\quad + E[(W+D(0, L_0))^2] + Q E[W+D(0, L_0)] \\ &\quad + \frac{Q^2}{3} \end{aligned} \quad (6.38)$$

The only information still lacking are the first two moments of  $W$ . It has been conjectured that  $T_U$  is homogeneously distributed on  $(0, R)$ . Therefore

$$E[N(R-T_U)] = \frac{1}{R} \int_0^R E[N_A(t)] dt \quad (6.39)$$

$$E(N^2(R-T_U)) = \frac{1}{R} \int_0^R E[N_A^2(t)] dt, \quad (6.40)$$

where  $N_A(\cdot)$  is the renewal process associated with  $\{A_n\}$ .

Application of renewal theoretic results reveals that (cf. (2.26) and (2.27))

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_0^x E[N(t)] dt - \left[ \frac{x^2}{2E[A]} + \left( \frac{E[A^2]}{2E[A]} - 1 \right) x \right. \\ \left. + \frac{E^2[A^2]}{4E^3[A]} - \frac{E[A^3]}{6E^2[A]} \right] = 0 \end{aligned} \quad (6.41)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_0^x E[N^2(t)] dt - \left[ \frac{x^3}{3E^2[A]} + \left( \frac{E[A^2]}{E^3[A]} - \frac{3}{2E[A]} \right) x^2 \right. \\ \left. + \left( \frac{3E^2[A^2]}{2E^4[A]} - \frac{2E[A^3]}{3E^3[A]} - \frac{3E[A^2]}{2E^2[A]} + 1 \right) x \right. \\ \left. + \frac{E[A^4]}{6E^3[A]} - \frac{E[A^2]E[A^3]}{E^4[A]} + \frac{E^3[A^2]}{E^5[A]} - \frac{E[A^3]}{2E^2[A]} - \frac{3E^2[A^2]}{4E^3[A]} \right] = 0 \end{aligned} \quad (6.42)$$

Assuming  $R \gg E[A]$  we find

$$\int_0^R E[N(t)] dt \approx \frac{R^2}{2E[A]} + \left( \frac{E[A^2]}{2E[A]} - 1 \right) R + \frac{E^2[A]}{4E^3[A]} - \frac{E[A^3]}{6E^2[A]} \quad (6.43)$$

$$\begin{aligned}
 \int_0^R E[N^2(t)] dt &= \left[ \frac{R^3}{3E^2[A]} + \left[ \frac{E[A^2]}{E^3[A]} - \frac{3}{2E[A]} \right] R^2 \right. \\
 &+ \left[ \frac{3E^2[A^2]}{2E^4[A]} - \frac{2E[A^3]}{3E^3[A]} - \frac{3E[A^2]}{2E^2[A]} + 1 \right] R \\
 &\left. + \frac{E[A^4]}{6E^3[A]} - \frac{E[A^2]E[A^3]}{E^4[A]} + \frac{E^3[A^2]}{E^5[A]} - \frac{E[A^3]}{2E^2[A]} - \frac{3E^2[A^2]}{4E^3[A]} \right] = 0
 \end{aligned} \tag{6.44}$$

and assuming gamma distributed interarrival times,

$$E[N(R-T_U)] \approx \frac{R}{2E[A]} + \frac{(c_A^2-1)}{2} + \frac{1}{R} \frac{(1-c_A^4)}{12} E[A] \tag{6.45}$$

$$E[N^2(R-T_U)] \approx \frac{R^2}{3E^2[A]} + \left( c_A^2 - \frac{1}{2} \right) \frac{R}{E[A]} + \frac{1}{6} (c_A^2-2)(c_A^2-1) - \frac{1}{12R} (1-c_A^4) E[A] \tag{6.46}$$

Once we know  $E[N(R-T_U)]$  and  $E[N^2(R)]$ , it is an easy matter to calculate  $E[W]$  and  $E[W^2]$  from

$$E[W] = E[N(R-T_U)] E[D] \tag{6.47}$$

$$E[W^2] = E[N(R-T_U)] \sigma^2(D) + E[N^2(R-T_U)] E^2[D] \tag{6.48}$$

Note that the assumption of  $R \gg E[A]$  is not unrealistic. Indeed, if we use a periodic review policy it does not make sense to have a review frequency higher than the arrival frequency. In that case reviews triggered by customer arrivals are more economic. In that case we use the standard (b,Q)-model.

This concludes the analysis of the service measures  $P_2$  and  $\hat{P}_1$ . For both measures we have derived approximations based on the PDF-method. It remains to validate the approximations. Results of the validation are given in chapter 8.

method. It remains to validate the approximations. Results of the validation are given in chapter 8.

### 6.3. Physical stock and backlog

As has been shown in the preceding chapters the mean physical stock depends on the way the inventory transactions are processed. The discrete time model assumes batch processing of the inventory transactions. This implies that the administrative stock is constant during the day, say and updated daily. This also implies an overestimation of the actual stock. The smaller the time between inventory updates, the smaller the bias of the estimation. This situation is modelled in the discrete time model. Hence the discrete time model yields an overestimate of the physical stock.

The compound renewal case describes on line processing of inventory transactions. In that case the administrative stock equals the actual stock. Hence the mean physical stock is properly estimated by the continuous monitoring model.

As with the  $\hat{P}_1$ -measure we must distinguish between the discrete time model and the compound renewal model. For both models we derive approximate expressions based on renewal-theoretic results.

#### Case I: The discrete time model

For the discrete time model we can exploit results from chapter 2, which have already been used in chapter 3 for the (R,S)-model. More specifically, the starting point for our analysis is the function  $K(x,t)$  defined as

$H(x,t) \quad := \quad$  the expected surface between the net stock and the zero level during  $(0,t]$ , given that at time 0 the net stock equals  $x \geq 0$ .

Note that  $t$  should be a multiple of the time unit.

The function  $H(x,t)$  has been studied in chapter 2 and equation (2.56) with  $E[A]$  identical to one time unit tells us that

$$K(x, t) = \int_0^x (x-y) dM(y) - \int_0^x \int_0^{x-y} (x-y-z) dM(z) dF_{D(0,t]}(y) \quad (6.49)$$

Then by conditioning on the net stock at time  $L_0$  we find that

$$E[X^*(b, Q)] \approx \frac{1}{E[\sigma_1]} \int_0^\infty \int_0^{b+Q} K(b+Q-y, t) dF_{D(0,L_0]+U_{0,x}}(y) dF_{(\sigma_1, L_1-L_0]}(t) \quad (6.50)$$

Substitution of (6.49) in (6.50) and some algebra yields

$$E[X^*(b, Q)] \approx \frac{1}{E[\sigma_1]} \left\{ \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z) dM(z) dF_{D(0,L_0]+U_{0,x}}(y) - \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z) dM(z) dF_{D(0,\sigma_1,L_1]+U_{0,x}}(y) \right\}$$

As in the analysis preceding equation (6.25) we note that

$$U_{0,R} = U_0 + W$$

with  $W$  defined below (6.23) and

$$D(0, \sigma_1+L_1] + U_{0,R} = Q + D(\sigma_1, \sigma_1+L_1] + U_{1,R}$$

Substituting these results into the above approximation for  $E[X^*(b, Q)]$  yields



$$E[X^+(b, Q)] \approx \frac{1}{E[\sigma_1]} \left\{ \int_0^{b+Q} \frac{(b+Q-y)^2}{2E[D]} dF_{W+D(0, L_0)}(y) - \int_0^b \frac{(b-y)^2}{2E[D]} dF_{W+D(0, L_0)}(y) \right\}$$

Using  $E[\sigma_1] = Q/E[D]$  we find after some algebra

$$E[X^+(b, Q)] \approx b + \frac{Q}{2} - E[W+D(0, L_0)] + \frac{1}{Q} \left\{ \int_0^\infty \frac{(y-b)^2}{2} dF_{W+D(0, L_0)}(y) - \int_{b+Q}^\infty \frac{(y-(b+Q))^2}{2} dF_{W+D(0, L_0)}(y) \right\} \quad (6.51)$$

Equation (6.51) is by now standard for further evaluation. Before doing so we relate  $E[X^+(b, Q)]$  to  $E[B(b, Q)]$ , the average backlog. This relation has already been derived in chapter 3. We repeat the arguments here for the reader's convenience.

Assume the stock keeping facility pays the supplier \$1 per purchased product per time unit this product is on order with the supplier. Then per order on average \$  $E[L].Q$  is paid, assuming  $Q$  is large compared to the undershoot of the reorder level  $b$ . Since on average every  $Q/E[D]$  time units a batch of  $Q$  products is ordered at the supplier, the average payment per unit time equals

$$E[L].Q / (Q/E[D]) = E[D]E[L].$$

On the other hand, the supplier receives on average \$  $E[O]$  per time unit, where

$E[O] :=$  the average amount on order.

Therefore

$$E[O] = E[D]E[L].$$

The basic equation determining the inventory position tells us that

$$E[Y] = E[X^*(b, Q)] + E[O] - E[B(b, Q)]$$

and thus

$$E[B(b, Q)] = E[X^*(b, Q)] + E[D]E[L] - E[Y]$$

We need an expression for  $E[Y]$ . From the analysis in Hadley and Whitin [1963] it can be derived that

*the inventory position at review moments is homogeneously distributed between  $b$  and  $b+Q$ .*

Consider an arbitrary review cycle  $(0, R)$ . At time 0 the inventory position equals  $x$ . Then it follows from the expression for the complementary holding cost given by (2.67) that the average inventory position during a review cycle with initial inventory position  $x$  equals  $x - \frac{1}{2}(R-1)E[D]$ . Conditioning on the homogeneously distributed initial inventory position yields

$$E[Y] = b + \frac{1}{2}Q - \frac{1}{2}(R-1)E[D]$$

This finally yields

$$E[B(b, Q)] = E[X^*(b, Q)] + E[D]E[L] - b - \frac{1}{2}Q + \frac{1}{2}(R-1)E[D] \quad (6.52)$$

Let us now reconsider (6.51). We know that

$$E[D(0, L_0)] = E[L] E[D]$$

$$E[W] = \frac{1}{2} (R-1) E[D]$$

and thereby

$$\begin{aligned} E[X^+(b, Q)] &= b + \frac{Q}{2} - \frac{1}{2} R(-1) E[D] - E[L] E[D] \\ &+ \frac{1}{Q} \left\{ \int_b^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0, L_0)}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0, L_0)}(y) \right\} \end{aligned} \quad (6.53)$$

Then it follows from (6.52) and (6.53) that

$$\begin{aligned} E[B(b, Q)] &= \frac{1}{Q} \left\{ \int_b^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0, L_0)}(y) \right. \\ &\quad \left. - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0, L_0)}(y) \right\} \end{aligned} \quad (6.54)$$

For the case of  $b < -Q$  we directly obtain

$$E[B(b, Q)] = E[W+D(0, L_0)] - b - \frac{Q}{2} \quad b < -Q \quad (6.55)$$

$$E[B(-Q, Q)] = \frac{Q}{2} + E[W+D(0, L_0)] \quad (6.56)$$

Define  $\gamma(\cdot)$  by

$$\gamma(x) = 1 - \frac{E[B(x-Q, Q)]}{E[B(-Q, Q)]} \quad x \geq 0 \quad (6.57)$$

Then  $\gamma(\cdot)$  is a probability distribution function. Let  $X_\gamma$  be the random variable which has a gamma distribution  $\hat{\gamma}(\cdot)$  with the same first two moments as  $\gamma(\cdot)$ . Then

$$E[X_\gamma] = \frac{\frac{Q^2}{6} + E\left[\frac{W+D(0, L_0)}{2}\right]Q + \frac{E[(W+D(0, L_0))^2]}{2}}{\frac{Q}{2} + E[W+D(0, L_0)]} \quad (6.58)$$

$$E[X_\gamma^2] = \left\{ \frac{Q^3}{12} + \frac{E[W+D(0, L_0)]}{3}Q^2 + E[(W+D(0, L_0))^2] \frac{Q}{2} + \frac{E[(W+D(0, L_0))^3]}{3} \right\} / \left\{ \frac{Q}{2} + E[W+D(0, L_0)] \right\} \quad (6.59)$$

Once we determined  $\hat{\gamma}(\cdot)$  from (6.58) and (6.59) we can approximate  $E[B(b, Q)]$  and  $E[X^+(b, Q)]$  by

$$\begin{aligned} E[B(b, Q)] &\approx \left( \frac{Q}{2} + E[W+D(0, L_0)] \right) (1 - \hat{\gamma}(b+Q)) & b \geq -Q \\ &- b - \frac{Q}{2} + E[W+D(0, L_0)] & b < -Q \end{aligned} \quad (6.60)$$

$$\begin{aligned} E[X^+(b, Q)] &= b+Q - \left( \frac{Q}{2} + E[W+D(0, L_0)] \right) \hat{\gamma}(b+Q) & b \geq -Q \\ &0 & b < -Q \end{aligned} \quad (6.61)$$

#### Case II: The compound renewal model

As in the case of an arrival process with constant interarrival times our starting point for our analysis is an expression for the function  $H(x, t)$ . For the present case of a compound renewal arrival process an approximation for  $H(x, t)$  is given by (cf. 2)

$$\begin{aligned}
 H(x, t) \approx & (E[\tilde{A}] - E[A]) \left[ x - \int_0^x (x-y) dF_{D(0,t]}(y) \right] \\
 & + E[A] \left[ \int_0^x (x-y) dM(y) - \int_0^x \int_0^{x-y} (x-y-z) dM(z) dF_{D(0,t]}(y) \right]
 \end{aligned} \quad (6.62)$$

We condition on the net stock at the start of the replenishment cycle  $(L_0, \sigma_1 + L_1]$ , leading to an expression for  $E[X^+(b, Q)]$ ,

$$E[X^+(b, Q)] = \frac{1}{E[\sigma_1]} \int_0^\infty \int_0^{b+Q} H(b+Q-y, t) dF_{U_{0,x}+D(0,L_0]}(y) dF_{\sigma_1+L_1-L_0}(t) \quad (6.63)$$

We substitute (6.62) into (6.63) and after application of some probabilistic arguments we obtain

$$\begin{aligned}
 E[X^+(b, Q)] \approx & \frac{1}{E[\sigma_1]} \left\{ (E[\tilde{A}] - E[A]) \left[ \int_0^{b+Q} (b+Q-y) dF_{U_{0,x}+D(0,L_0]}(y) \right. \right. \\
 & \left. \left. - \int_0^{b+Q} (b+Q-y) dF_{U_{0,x}+D(0,\sigma_1+L_1]}(y) \right] \right. \\
 & + E[A] \left[ \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z) dM(z) dF_{U_{0,x}+D(0,L_0]}(y) \right. \\
 & \left. \left. - \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z) dM(z) dF_{U_{0,x}+D(0,\sigma_1+L_1]}(y) \right] \right\}
 \end{aligned}$$

Applying the by now standard arguments concerning  $D(0, \sigma_1 + L_1]$  and  $U_{0,R}$  we find after some algebra



$$\begin{aligned}
 E[X^+(b, Q)] &\approx \frac{(c_A^2 - 1)}{2} \frac{E[A]}{Q} \left\{ \int_0^{b+Q} (b+Q-y) dF_{U_{0,x}+D(0,L_0)}(y) \right. \\
 &\quad \left. - \int_0^b (b-y) dF_{U_{0,x}+D(0,L_0)}(y) \right\} \\
 &+ \frac{E[D]}{Q} \left[ \int_0^{b+Q} \frac{(b+Q-y)^2}{2E[D]} dF_{W+D(0,L_0)}(y) \right. \\
 &\quad \left. - \int_0^b \frac{(b-y)^2}{2E[D]} dF_{W+D(0,L_0)}(y) \right]
 \end{aligned} \tag{6.64}$$

where  $W$  is defined in section 2, when deriving an expression for the  $\hat{P}_1$ -measure.

The second term on the right hand side of (6.64) is identical to the expression for  $E[X^+(b, Q)]$  for the discrete time case given above by equation (6.51). Hence we apply the same transformation rules. The first term on the right hand side of (6.64) can also be rewritten by writing the integral from 0 to  $b+Q$  as the difference between the integral from 0 to  $\infty$  and the integral from  $b+Q$  to  $\infty$ . This yields

$$\begin{aligned}
 E[X^+(b, Q)] &\approx \frac{(c_A^2 - 1)}{2} E[D] - \frac{(c_A^2 - 1)}{2} \frac{E[D]}{Q} \left\{ \int_b^\infty (y-b) dF_{U_{0,x}+D(0,L_0)}(y) \right. \\
 &\quad \left. - \int_{b+Q}^\infty (y-(b+Q)) dF_{U_{0,x}+D(0,L_0)}(y) \right\} \\
 &+ b + \frac{Q}{2} - E[W+D(0, L_0)] \\
 &+ \frac{1}{Q} \left\{ \int_b^\infty \frac{(y-b)^2}{2} dF_{W+D(0,L_0)}(y) - \int_{b+Q}^\infty \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0)}(y) \right\}
 \end{aligned} \tag{6.65}$$

Instead of fitting distribution to  $U_{0,R}+D(0,L_0]$  and  $W+D(0,L_0]$  explicitly calculating the integrals, we apply the PDF-method to the mean backlog. This can be done since we have an explicit relation between the mean physical stock and the mean backlog.

$$E[X^*(b,Q)] = E[Y(b,Q)] - E[D] \frac{E[L]}{E[A]} + E[B(b,Q)] \quad (6.66)$$

Equation (6.66) has been derived in exactly the same way as its equivalent in the discrete time model.

To obtain an expression for  $E[Y(b,Q)]$  we consider an arbitrary review cycle  $(0,R)$ . We assume that review moments are arbitrary moments in time from the point of view of the arrival process. We further assume that at time 0 the inventory position equals  $x$ . To calculate the average inventory position in  $(0,R)$ , we divide the expected area between the  $x$ -level and the inventory position by  $R$  and subtract this from  $x$ . In chapter 2 we have already analyzed this expected area and found that this is approximately equal to  $\frac{1}{2}R^2E[D]/E[A]$ . Hence the average inventory position in  $(0,R)$  equals  $x - \frac{1}{2}R^2/E[A]E[D]$ . Now it follows from the analysis in Hadley and Whitin that the inventory position at the beginning of an arbitrary review cycle is homogeneously distributed on  $(b,b+Q)$ . This yields

$$E[Y(b,Q)] \approx b + \frac{1}{2}Q - \frac{R}{2E[A]}E[D]$$

and so

$$\begin{aligned} E[X^*(b,Q)] &\approx b + \frac{Q}{2} - \frac{R}{2E[A]}E[D] - E[D] \frac{E[L]}{E[A]} \\ &\quad + E[B(b,Q)] \end{aligned} \quad (6.67)$$

We reconsider (6.65). It follows from (6.45) and (6.47) that

$$E[W] \approx \left[ \frac{R}{2E[A]} + \frac{(c_A^2-1)}{2} + \frac{(1-c_A^4)}{12R} E[A] \right] E[D]$$

From (6.17) we know that

$$E[D(0, L_0)] \approx \frac{E[L]}{E[A]} E[D]$$

Then (6.65) becomes

$$\begin{aligned} E[X^*(b, Q)] &= b + \frac{Q}{2} - \frac{E[L]}{E[A]} E[D] - \frac{R}{2E[A]} E[D] - \frac{(1-c_A^4)}{12} \frac{E[A]}{R} E[D] \\ &\quad - \frac{(c_A^2-1)}{2} \frac{E[D]}{Q} \left\{ \int_b^\infty (Y-b) dF_{U_{0,x}+D(0,L_0)}(Y) \right. \\ &\quad \left. - \int_{b+Q}^\infty (Y-(b+Q)) dF_{U_{0,x}+D(0,L_0)}(Y) \right\} \\ &\quad + \frac{1}{Q} \left\{ \int_b^\infty \frac{(Y-b)^2}{2} dF_{W+D(0,L_0)}(Y) - \int_{b+Q}^\infty \frac{(Y-(b+Q))^2}{2} dF_{N+D(0,L_0)}(Y) \right\} \end{aligned} \quad (6.68)$$

Comparison of (6.67) and (6.68) suggest that

$$\begin{aligned}
 E[B(b, Q)] = & \frac{(1-C_A^4)}{12} \frac{E[A]}{R} E[D] - \frac{(C_A^2-1)}{2} \frac{E[D]}{Q} \left\{ \int_b^{\infty} (y-b) dF_{U_{0,x}+D(0,L_0)}(y) \right. \\
 & \left. - \int_{b+Q}^{\infty} (y-(b+Q)) dF_{U_{0,x}+D(0,L_0)}(y) \right\} \\
 & + \frac{1}{Q} \left\{ \int_b^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0,L_0)}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0)}(y) \right\}
 \end{aligned}$$

This, however, is inconsistent with  $\lim E[B(b, Q)] = 0$ . This inconsistency is caused by the approximations for  $E[W]$  and  $H(x, t)$ . On the other hand, assuming that  $R \gg E[A]$  we may assume that

$$\frac{(1-C_A^4)}{12} \frac{E[A]}{R} \text{ negligible.}$$

Therefore we suggest to approximate the mean backlog by

$$\begin{aligned}
 E[B(b, Q)] = & \frac{1}{Q} \left\{ \int_b^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0,L_0)}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0)}(y) \right\} \\
 & - \frac{(C_A^2-1)}{2} \frac{E[D]}{Q} \left\{ \int_b^{\infty} (y-b) dF_{U_{0,x}+D(0,L_0)}(y) - \int_{b+Q}^{\infty} (y-(b+Q)) dF_{U_{0,x}+D(0,L_0)}(y) \right\} \quad (6.69)
 \end{aligned}$$

It follows from (6.69) (as well as from (6.68)), that

$$E[B(b, Q)] = E[W+D(0, L_0)] - \frac{Q}{2} - b - \frac{(C_A^2-1)}{2} E[D] \quad b \leq -Q \quad (6.70)$$

An expression for  $E[B(b, Q)]$  for  $b \geq -Q$  is derived from application of the PDF-method.

Let  $\gamma(\cdot)$  be the pdf defined by

$$\gamma(x) = 1 - \frac{E[B(x-Q, Q)]}{E[B(-Q, Q)]}, \quad x \geq 0$$

Let  $X_\gamma$  be the random variable  $X_\gamma$  with pdf  $\gamma(\cdot)$ . Then the first two moments of  $X_\gamma$  are given by

$$\begin{aligned} E[X_\gamma] = & \left\{ \frac{Q^2}{6} + \frac{E[W+D(0, L_0)]}{2} Q + \frac{E[(W+D(0, L_0))^2]}{2} \right. \\ & \left. - \frac{(c_A^2-1)}{2} E[D] \left( \frac{Q}{2} + E[W+D(0, L_0)] \right) \right\} / \\ & \left\{ \frac{Q}{2} + E[W+D(0, L_0)] - \frac{(c_A^2-1)}{2} E[D] \right\} \end{aligned} \quad (6.71)$$

$$\begin{aligned} E[X_\gamma^2] = & \left\{ \frac{Q^3}{12} + \frac{E[W+D(0, L_0)]}{3} Q^2 + E[(W+D(0, L_0))^2] \frac{Q}{2} \right. \\ & + \frac{E[(W+D(0, L_0))^3]}{3} - \frac{(c_A^2-1)}{2} E[D] \left[ \frac{Q^2}{3} + E[W+D(0, L_0)] Q \right. \\ & \left. \left. + E[(W+D(0, L_0))^2] \right] \right\} \\ & \left\{ \frac{Q}{2} + E[W+D(0, L_0)] - \frac{(c_A^2-1)}{2} E[D] \right\} \end{aligned} \quad (6.72)$$

Fitting the gamma distribution  $\hat{\gamma}(\cdot)$  to  $E[X_\gamma]$  and  $E[X_\gamma^2]$  we have the following approximation for  $E[B(b, Q)]$  for  $b \geq -Q$ ,

$$E[B(b, Q)] \approx \left( \frac{Q}{2} + E[W+D(0, L_0)] \right) (1 - \hat{\gamma}(b+Q)), \quad b \geq -Q \quad (6.73)$$

Substituting this approximation into (6.67) yields

$$E[X^*(b, Q)] \approx b + \frac{Q}{2} - \frac{R}{2E[A]} E[D] - E[D] \frac{E[L]}{E[A]} + \left( \frac{Q}{2} + E[W+D(0, L_0)] \right) (1 - \hat{\gamma}(b+Q)) , \quad b \geq -Q \quad (6.74)$$

This completes the analyses of the (R,b,Q)-model. The analysis turned out to be quite similar to that of the (b,Q)-model. Main differences are caused by the undershoots of b during the review period, which leads to the introduction of the random variable W.



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